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COMPLEX MATRICES AND THEIR PROPERTIES

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ABSTRACT

By this paper, our aim is to introduce the Complex Matrices that why we require the complex matrices and we have discussed about the different types of complex matrices and their properties.

I. INTRODUCTION

It is no longer possible to work only with real matrices and real vectors. When the basic problem was Ax = b the solution was real when A and b are real. Complex number could have been permitted, but would have contributed nothing new. Now we cannot avoid them. A real matrix has real coefficients in det (A - λI), but the eigen values may complex.

We now introduce the space C^n of vectors with n complex components. The old way, the vector in C^2 with components (l, i) would have zero length: $1^2 + i^2 = 0$ not good.

The correct length squared is $1^2 + 1i1^2 = 2$

This change to $11x11^2 = 1x_1 1^2 + \ldots\ldots\ldots |x_n|^2$ forces a whole series of other changes. The inner product, the transpose, the definitions of symmetric and orthogonal matrices all need to be modified for complex numbers.

II. DEFINITION

A matrix whose elements may contain complex numbers called complex matrix.

The matrix product of two 2×2 complex matrices is given by

$$\begin{bmatrix} x_{11}+y_{11}\,i & x_{12}+y_{12}\,i \\ x_{21}+y_{21}\,i & x_{22}+y_{22}\,i \end{bmatrix}\begin{bmatrix} u_{11}+v_{11}\,i & u_{12}+v_{12}\,i \\ u_{21}+v_{21}\,i & u_{22}+v_{22}\,i \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} + i\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix},$$

where

$$\begin{aligned}
R_{11} &= u_{11}\,x_{11} + u_{21}\,x_{12} - v_{11}\,y_{11} - v_{21}\,y_{12} \\
R_{12} &= u_{12}\,x_{11} + u_{22}\,x_{12} - v_{12}\,y_{11} - v_{22}\,y_{12} \\
R_{21} &= u_{11}\,x_{21} + u_{21}\,x_{22} - v_{11}\,y_{21} - v_{21}\,y_{22} \\
R_{22} &= u_{12}\,x_{21} + u_{22}\,x_{22} - v_{12}\,y_{21} - v_{22}\,y_{22} \\
I_{11} &= v_{11}\,x_{11} + v_{21}\,x_{12} + u_{11}\,y_{11} + u_{21}\,y_{12} \\
I_{12} &= v_{12}\,x_{11} + v_{22}\,x_{12} + u_{12}\,y_{11} + u_{22}\,y_{12} \\
I_{21} &= v_{11}\,x_{21} + v_{21}\,x_{22} + u_{11}\,y_{21} + u_{21}\,y_{22} \\
I_{22} &= v_{12}\,x_{21} + v_{22}\,x_{22} + u_{12}\,y_{21} + u_{22}\,y_{22}.
\end{aligned}$$

III. LENGTHS AND TRANSPOSES IN THE COMPLEX CASE

The complex vector space C^n contains all vectors x with n complex components.

Complex Vector x = x_1 with components $x_j = a_j + i\,b_j$
x_2
x_n

Vectors x and y are still added component by component. Scalar multiplication cx is now done with complex numbers c. The vectors V_1, V_2 ….. V_k are linearly dependent if some non trivial combination gives $C_1 V_1 + C_2$

$V_2 + \ldots\ldots C_k V_k = 0$; the C_j may now be complex. The unit Co-ordinate vectors are still in C^n, they are still independent, and they still form a basis. Therefore C^n is a complex vector space of dimension n. In case of length, each X_j^2 is replaced by its modulus $1x_j 1^2$

Length squared $11x11^2 = 1x_1 1^2 + \ldots.. 1x_n1^2$

For real vectors there was a close connection between the length and inner product $11x11^2 = x^T x$

Replacing x by $\bar{x}$, the inner product becomes.

$$\bar{x}^T y = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \ldots\ldots \bar{x}_n y_n$$

IV. HERMITIAN MATRICES

With complex entries the idea os symmetric matrices has to be extended. The matrices that equal their conjugate transpose are Hermitian matrices.

"A Hermitian" $A^H = \bar{A}^T = A^K$

The diagonal entries must be real; they are unchanged by conjugation. Each off diagonal entry is matched with its mirror image across the main diagonal and $a_{ij} = \bar{a}_{ji}$

There are three basic properties of Hermitian Matrices:

1. If $A = A^H$ then for all complex vectors x, the number is real.
2. If $A = A^H$ then every eigen value is real.
3. Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigen values are orthogonal to one another.

The diagonalizing matrix can be chosen with orthonormal columns when $A = A^H$

In case A is real and symmetric, its eigenvalues are real by property. (2). Its unit eigenvectors are orthogonal by property (3). Those eigenvectors are also real; they solve $(A-\lambda I)\, x = 0$ These orthonormal eigenvectors go into an orthogonal matrix Q, with $Q^T Q = I$ and $Q^T = Q^{-1}$. Then $S^{-1} AS = \Lambda$ becomes special - it is $Q^{-1} AQ = \wedge$ or $A = Q\wedge Q^{-1}$

Now we can state one of the great theorms of Linear algebra: A real symmetric matrix can be factored into $A=Q\wedge Q^T$. Its orthonormal eigenvectors are in the orthogonal matrix Q and its eigenvalues are in $\wedge$ where $\wedge = \mathrm{diag}\,(\lambda_1,\lambda_2.....)$

If we multiply columns by rows, the matrix A becomes a combination of one dimensional projections which are the special xx^T of rank 1, multipled by λ

$$A = Q\wedge Q^T =$$
$$= \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \text{-------} \lambda_n x_n x_n^T$$

If A is real, all complex eigenvalues come in conjugate pairs : $Ax = \lambda x$ and $A\bar{x} = \bar{\lambda}\bar{x}$ If a+ib is an eigen value of a real matrix so is a - ib.

V. UNITARY MATRICES

A complex matrix with orthonormal columns is called a Unitary matrix. A unitary matrix can be compared to a number on the unit circle a complex number of absolute value 1.

Unitary Matrix $U^H U = I$, $UU^H = I$ and $U^H = U^{-1}$

There are three properties

1. $(Ux)^H (Uy) = x^H U^H Uy = x^H y$ and lengths are preserved by
   U : $11Ux\,11^2 = x^H U^H Ux = 11x11^2$
2. Every eigenvalue of U has absolute value $1\lambda 1 = 1$.
   This follows directly from $Ux = \lambda x$, by comparing the lengths of the two sides $11Ux11 = 11x11$ and always $11\lambda x11 = 1\lambda 111x11$. therefore $I\lambda I = 1$.

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3. Eigenvectors corresponding to different eigenvalues are orthonomal start with Ux = λ_1x and Uy = λ_2y and take inner product $x^H y = (Ux)^H (Uy) = (\lambda_1 x)^H (\lambda_2 y) = \bar{\lambda}_1 \lambda_2 x^H y \Rightarrow \bar{\lambda}_1 \lambda_2 = 1$ or $x^H y = 0$ But from property (2) $\bar{\lambda}_1 \lambda = 1$ so we can not also have $\lambda_1 \lambda_2 = 1$. Thus $x^H y = 0$ and the eigenvectors are orthogonal.

VI. SKEW HERMITIAN MATRICES

A square matrix with complex entries is said to be skew Hermitian or antihermitian if its conjugate transpose is equal to its negative. Thus skew Hermitian matrices satisfy $k^H = -k$ just as skew symmetric matrices satisfy $k^T = -k$ Where H denotes the conjugate transpose of a matrix.

VII. PROPERTIES

Their properties follow immediately from their close link to Hermitian Matrices :

1. The eigen values of a skew Hermitian matrix are all purely imaginary or zero. Further more, skew Hermitian matrices are normal. Hence they are diagonalizable and their eigen vectors for distinct eigen values must of orthogonal. Thus we still have K = U^UH with a unitary U instead of a real orthogonal Q.
2. All entries on main doagonal of Skew Hermitian matrix have to be pure imaginary.
3. If A is skew Hermitian, then iA is Hermitian.
4. If A is skew Hermitian then AK is Hermitian if K is an even integar and skew Hermitian if K is an odd integer.

VIII. NORMAL MATRIX

A complex square matrix A is normal if $A^H A = AA^H$ where H is the conjugate transpose. That is, a matrix is normal if it commutes with its conjugate transpose. A matrix with real entries satisfies $A^H = A^T$ and therefore normal if $A^T A = AA^T$

Normality is a convenient test for diagonalizability : a matrix is normal if and only if it is unitary similar to a diagonal matrix and therefore any matrix satisfying the equation $A^H A = AA^H$ is diagonalizable.

The concept of normal matrices can be extended to normal operators on infinite dimensional Hilbert spaces and to normal elements in c^n- algebra.

Among complex matrices, all unitary, Hermitian and skew Hermitian matrices are normal. Likewise among real matrices, all orthogonal, symmetric and skew symmetric matrices are normal.

IX. CONSEQUENCES

The concept of normality is important because normal matrices are precisely those to which the spectral theorm applies ; a matrix is normal if and only if it can be represented by a diagonal matrix ^ and a unitary matrix U by the formula.

The entries λ of the diagonal matrix ^ are the eigen values of A, and the columns of U are the eigen vectors of A.

X. EQUIVALENT DEFINITIONS:-

It is possible to give a fairly long list of equivalent definitions of a normal matrix. Let A be a n x n complex matrix. Then

1. A is normal
2. A is diagonalizable by a unitary matrix.
3. Then entire space is spanned by some orthonormal set of eigen vectors of A.
4. 11 Ax 11 = 11 A^Hx 11 for every x
5. $tr(A^H A) = \sum_{j=1}^{n} 1\lambda j\, 1^2$
6. A^H is a polynomial (of degree ≤ n-1) in A.
7. A^H = AU for some unitary matrix U.
8. U and P commute where we have the polar decomposition A = UP with a unitary matrix U and some positive semidefinite matrix P.
9. A commutes with some normal matrix N with distinct eigenvalues.

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XI. CONCLUSION

Thus we can summerized by a table of paralles between Real and Complex.

XII. REAL VERSUS COMPLEX

R^n (n real components)	C^n (n Complex Components)
Length: $11x11^2 = x_1^2 + x_2^2 \ldots\ldots x_n^2$	Length: $11x11^2 = 1x_11^2 + 1x_21^2 + \ldots\ 1x_n1^2$
Transpose: $A^T{}_{ij} = A_{ji}$	Hermitian Transpose: $A^H{}_{ij} = \bar{A}_{ji}$
$(AB)^T = B^TA^T$	$(AB)^H = B^H A^H$
Inner Product: $x^Ty = x_1y_1 + \ldots\ldots\ x_ny_n$	$x^Hy = \bar{x}_1y_1 + \ldots\ldots\ \bar{x}_ny_n$
$(Ax)^T y = x^T (A^Ty)$	$(Ax)^Hy = x^H (A^Hy)$
Orthoganility: $x^Ty = 0$	Orthoganility: $x^Hy = 0$
Symmetric matrices: $A^T = A$	Hermitian matrices: $A^H = A$
$A = Q \wedge Q^{-1} = Q \wedge Q^T$	$A = U \wedge U^{-1} = U \wedge U^H$
(Real $\wedge$)	(Real $\wedge$)
Skew Symmetric: $k^T = -k$	Skew Hermitian: $k^H = -k$
Orthoganal: $Q^TQ = I = QQ^T$	Unitary: $U^HU = I = UU^H$
or $Q^T = Q^{-1}$	or $U^H = U^{-1}$
$(Qx)^T (Q y) = x^Ty$ and	$(U x)^H (U y) = x^Hy$ and
$11Qx11 = 11x11$	$11Ux11 = 11x11$

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